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The general solutions to the reflection equation of the Izergin–Korepin model

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Abstract. We obtain the general solutions to the reflection equation of the Izergin–Korepin model. The general solutions have two free parameters and will reduce to the non-trivial diagonal solutions when both free parameters vanish. It will also reduce to the solutions with upper–lower triangular structures when one of the parameters vanishes. Moreover, the Hamiltonian with boundary terms for the system is obtained.

1. Introduction

Since Sklyanin proposed the generalized algebra Bethe ansatz method to deal with the open-boundary solvable models based on the reflection equation [1, 2], much work has been done on the solutions of the reflection equation [3–16]. However, at present, there are only a few models whose general reflecting K-matrices have been obtained [11–16].

The R-matrix of the Izergin–Korepin model [17] or $A_2^{(2)}$ model [18, 19] is the simplest example of an R-matrix of the twisted type. The diagonal solutions to the reflection equation for the model were first obtained by Mezincescu and Nepomechie [8] by solving the reflection equation directly, and later obtained by Batchelor *et al* [9] by taking the vertex limit to the diagonal face-reflecting K-matrices of the $A_2^{(2)}$ model. The non-diagonal case was first considered by Kim [10]. By assuming the solutions to be proportional to identity when the spectral parameter u is zero, he obtained three families of non-diagonal solutions. One family of solutions will reduce to the identity solution when one of the parameters in the solutions vanishes. The other two families of solutions have only one free parameter and have upper–lower triangular structures. However, no solutions with every element of the K-matrix non-vanishing are obtained.

Enlightened by the work of Inami *et al* in calculating the general reflecting K-matrices for the $A_1^{(1)}$ model [16], we present the general reflecting K-matrices for the $A_2^{(2)}$ model by solving the reflection equation directly. The general solutions we obtained have two free parameters and will reduce to the non-trivial diagonal solutions when both free parameters vanish. It will also reduce to the solutions with upper–lower triangular structures when one of the parameters vanishes, which include two cases solutions in [10]. Moreover, the Hamiltonian with general boundary terms for the model is obtained.

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2. The Izergin–Korepin model and reflection equation

The R-matrix for the Izergin–Korepin model [17] or $A_2^{(2)}$ model [18, 19] is

$$R(u) = \begin{bmatrix} c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & g & 0 & f & 0 & 0 \\ 0 & \bar{e} & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{g} & 0 & a & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & e & 0 \\ 0 & 0 & \bar{f} & 0 & \bar{g} & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{e} & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \end{bmatrix} \tag{1}$$

with

$$\begin{aligned} a(u) &= \sinh(u-3q) - \sinh(5q) + \sinh(3q) + \sinh(q) \\ b(u) &= \sinh(u-3q) + \sinh(3q) \\ c(u) &= \sinh(u-5q) + \sinh(q) \\ d(u) &= \sinh(u-q) + \sinh(q) \\ e(u) &= -2\mathrm{e}^{-u/2}\sinh(2q)\cosh(u/2-3q) \\ \bar{e}(u) &= -2\mathrm{e}^{u/2}\sinh(2q)\cosh(u/2-3q) \\ f(u) &= -2\mathrm{e}^{-u+2q}\sinh(q)\sinh(2q) - \mathrm{e}^{-q}\sinh(4q) \\ \bar{f}(u) &= 2\mathrm{e}^{u-2q}\sinh(q)\sinh(2q) - \mathrm{e}^{q}\sinh(4q) \\ g(u) &= 2\mathrm{e}^{-u/2+2q}\sinh(u/2)\sinh(2q) \\ \bar{g}(u) &= -2\mathrm{e}^{u/2-2q}\sinh(u/2)\sinh(2q). \end{aligned} \tag{2}$$

One can check that the R-matrix satisfies the following properties:

$$\begin{aligned} &\text{regularity:} && R_{12}(0) = \rho(0)^{1/2}\mathcal{P}_{12} \\ &\text{unitarity:} && R_{12}(u)R_{12}^{t_1t_2}(-u) = \rho(u) \\ &\text{PT-symmetry:} && \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12} = R_{12}^{t_1t_2}(u) \\ &\text{crossing-symmetry:} && R_{12}(u) = \overset{1}{V} R_{12}^{t_2}(-u-\eta)\overset{1}{V}{}^{-1}. \end{aligned} \tag{3}$$

Here $\mathcal{P}$ is the exchange operator defined by $\mathcal{P}(x \otimes y) = y \otimes x$, t_i denotes transposition in the space i, $\overset{1}{V} = V \otimes 1$, $\overset{2}{V} = 1 \otimes V$, η is the crossing parameter and V determines the crossing matrix $M \equiv V^tV = M^t$ with $\eta = -6q - \sqrt{-1}\pi$ and $M = \mathrm{diag}(\mathrm{e}^{2q}, 1, \mathrm{e}^{-2q})$, $\rho(u) = ((\sinh(q) - \sinh(5q+u))(\sinh(q) - \sinh(5q-u)))$.

The R-matrix also fulfils the Yang–Baxter equation (YBE) [20, 21]

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \tag{4}$$

where $R_{12}(u)$, $R_{13}(u)$ and $R_{23}(u)$ act on $C^3 \otimes C^3 \otimes C^3$, with $R_{12}(u) = R(u) \otimes 1$, $R_{23}(u) = 1 \otimes R(u)$, etc.

For an $N \times N$ square lattice, if we can find $K_\pm(u)$ which satisfy the reflection equations given by [2, 22]

$$R_{12}(u-v)\overset{1}{K}_-(u)R_{12}^{t_1t_2}(u+v)\overset{2}{K}_-(v) = \overset{2}{K}_-(v)R_{12}(u+v)\overset{1}{K}_-(u)R_{12}^{t_1t_2}(u-v) \tag{5}$$

$$R_{12}(-u+v)\overset{1}{K}{}^{t_1}_{+}(u)\overset{1}{M}{}^{-1}R^{t_1t_2}_{12}(-u-v-2\eta)\overset{1}{M}\overset{2}{K}{}^{t_2}_{+}(v)$$
$$=\overset{2}{K}{}^{t_2}_{+}(v)\overset{1}{M}R_{12}(-u-v-2\eta)\overset{1}{M}{}^{-1}\overset{1}{K}{}^{t_1}_{+}(u)R^{t_1t_2}_{12}(-u+v) \quad (6)$$

where equation (5) is called the reflection equation and equation (6) is called the dual-reflection equation, $\overset{1}{K}_{\pm}(u)=K_{\pm}(u)\otimes 1$, $\overset{2}{K}_{\pm}(u)=1\otimes K_{\pm}(u)$. Then the transfer matrix $t(u)$ defined as

$$t(u)=\mathrm{tr}\,K_{+}(u)T(u)K_{-}(u)T^{-1}(-u) \quad (7)$$

can constitute a one-parameter commutative family $[t(u),t(v)]=0$. Here

$$T(u)=R_{01}(u)R_{02}\cdots R_{0N}(u) \quad (8)$$

the space V_0 is usually called the auxiliary space, the space $V_1\otimes V_2\cdots\otimes V_N$ is called the quantum space. The corresponding integrable open chain Hamiltonian takes the form

$$H=\sum_{k=1}^{N-1}H_{k,k+1}+\frac{1}{2}\overset{1}{K}{}'_{-}(0)+\frac{\mathrm{tr}\,\overset{0}{K}_{+}(0)H_{N,0}}{\mathrm{tr}\,K_{+}(0)} \quad (9)$$

where $H_{k,k+1}=\mathcal{P}_{k,k+1}R'_{kk+1}(u)|_{u=0}$.

From equations (5) and (6), one can see that, given a solution $K_{-}(u)$ of equation (5), the matrix

$$K_{+}(u)=K^{t}_{-}(-u-\eta)M \quad (10)$$

satisfies equation (6).

3. The solution to the reflection equation

After a tedious calculation (see the appendix), we achieve the $K_{-}(u)$ to equation (5) which is

$$K_{-}(u)=\rho^{K}(u)\begin{bmatrix} x_1(u) & y_{11}(u) & z_1(u)\\ y_{21}(u) & x_2(u) & y_{12}(u)\\ z_2(u) & y_{22}(u) & x_3(u)\end{bmatrix} \quad (11)$$

where $\rho^{K}(u)$ is an arbitrary function,

$$\begin{aligned} y_{11}(u)&=\mu_{-}(c-\mathrm{e}^{q-u})\sinh(u) & y_{21}(u)&=\tilde{\mu}_{-}(c-\mathrm{e}^{q-u})\sinh(u)\\ y_{12}(u)&=\mu_{-}\mathrm{e}^{-q}(c+\mathrm{e}^{u-q})\sinh(u) & y_{22}(u)&=\tilde{\mu}_{-}\mathrm{e}^{-q}(c+\mathrm{e}^{u-q})\sinh(u)\\ z_1(u)&=\mu_{-}^{2}\cosh(q-u)\sinh(u) & z_2(u)&=\tilde{\mu}_{-}^{2}\cosh(q-u)\sinh(u)\end{aligned} \quad (12)$$

$$\begin{aligned} x_1(u)&=\frac{\mathrm{e}^{-q-u}}{\cosh(q)}(c\cosh(q)+\sinh(u-2q))\\ &\quad+\tilde{\mu}_{-}\mu_{-}\cosh(q-u)(c\cosh(2q)-\mathrm{e}^{-u}\sinh(q))\\ x_2(u)&=\frac{\mathrm{e}^{-q}}{\cosh(q)}(c\cosh(q+u)-\sinh(2q))\\ &\quad+\tilde{\mu}_{-}\mu_{-}\cosh(q-u)(c\cosh(2q)+\sinh(u-q))\\ x_3(u)&=\frac{\mathrm{e}^{-q+u}}{\cosh(q)}(c\cosh(q)+\sinh(u-2q))\\ &\quad+\tilde{\mu}_{-}\mu_{-}\cosh(q-u)(c\cosh(2q)-\mathrm{e}^{u}\sinh(q))\end{aligned} \quad (13)$$

where μ_-, $\tilde{\mu}_-$ are arbitrary parameters, c satisfies $c^2 = -1$. If we choose

$$\rho^k(u) = (x_2(u)x_2(-u) + y_{11}(-u)y_{21}(u) + y_{12}(u)y_{22}(-u))^{-1/2} \tag{14}$$

there are

$$K_-(0) = 1 \qquad K_-(u)K_-(-u) = 1. \tag{15}$$

4. The Hamiltonian with boundary terms

Let $K_-(u) = K_-(u;\mu_-,\tilde{\mu}_-)$. We define $K_+(u) = K_-^t(-u-\eta;\tilde{\mu}_+,\mu_+)M$ due to (10),where $\mu_\pm$, $\tilde{\mu}_\pm$ are arbitrary parameters. From (1), (2), (9), (11)–(14), we obtain

$$\begin{aligned}
H = \sum_{n=1}^{N-1}\Big\{ & \tfrac{1}{4}a_1(s_n^+s_n^-s_{n+1}^+s_{n+1}^- + s_n^-s_n^+s_{n+1}^-s_{n+1}^+) + \tfrac{1}{4}a_2 s_n^+s_n^-s_{n+1}^-s_{n+1}^+ + \tfrac{1}{4}a_3 s_n^-s_n^+s_{n+1}^+s_{n+1}^- \\
& + a_4(s_n^3s_{n+1}^3 + (s_n^3)^2(s_{n+1}^3)^2) + \tfrac{1}{2}a_5(s_n^+s_n^-(s_{n+1}^3)^2 + (s_n^3)^2s_{n+1}^-s_{n+1}^+) \\
& + \tfrac{1}{2}a_6(s_n^-s_n^+(s_{n+1}^3)^2 + (s_n^3)^2s_{n+1}^+s_{n+1}^-) + \tfrac{1}{4}a_7((s_n^+)^2(s_{n+1}^-)^2 + (s_n^-)^2(s_{n+1}^+)^2) \\
& + \tfrac{1}{2}a_8(s_n^3s_n^+s_{n+1}^-s_{n+1}^3 + s_n^3s_n^-s_{n+1}^+s_{n+1}^3 + s_n^-s_n^3s_{n+1}^3s_{n+1}^+ + s_n^+s_n^3s_{n+1}^3s_{n+1}^+) \\
& - \tfrac{1}{2}a_9(s_n^3s_n^-s_{n+1}^3s_{n+1}^+ + s_n^+s_n^3s_{n+1}^-s_{n+1}^3) - \tfrac{1}{2}a_{10}(s_n^3s_n^+s_{n+1}^3s_{n+1}^- + s_n^-s_n^3s_{n+1}^+s_{n+1}^3)\Big\} \\
& + \frac{1}{\rho^k}\Bigg\{\tfrac{1}{2}x_1(s_1^3 + (s_1^3)^2) + \tfrac{1}{2}x_2(s_1^+s_1^- - s_1^3 - (s_1^3)^2) + \tfrac{1}{2}x_3((s_1^3)^2 - s_1^3) \\
& + \frac{y_1}{\sqrt{2}}(\mu_-s_1^3s_1^+ + \tilde{\mu}_-s_1^-s_1^3) - \frac{y_2}{\sqrt{2}}(\mu_-s_1^+s_1^3 + \tilde{\mu}_-s_1^3s_1^-) \\
& + \tfrac{1}{2}z(\mu_-^2(s_1^+)^2 + \tilde{\mu}_-^2(s_1^-)^2)\Bigg\} \\
& + \frac{1}{\operatorname{tr}\tilde{k}}\Bigg\{\frac{\tilde{x}_1}{2}(s_N^3 + (s_N^3)^2) + \frac{\tilde{x}_2}{2}(s_N^+s_N^- - s_N^3 - (s_N^3)^2) + \frac{\tilde{x}_3}{2}((s_N^3)^2 - s_N^3) \\
& + \frac{\tilde{y}_1}{\sqrt{2}}(\mu_+s_N^3s_N^+ + \mathrm{e}^{2q}\tilde{\mu}_+s_N^-s_N^3) - \frac{\tilde{y}_2}{\sqrt{2}}(\mu_+s_N^+s_N^3 + \mathrm{e}^{2q}\tilde{\mu}_+s_N^3s_N^-) \\
& + \tfrac{1}{2}\tilde{z}(\mathrm{e}^{-2q}\mu_+^2(s_N^+)^2 + \mathrm{e}^{2q}\tilde{\mu}_+^2(s_N^-)^2)\Bigg\} + \text{constant}\cdot id.
\end{aligned} \tag{16}$$

Here,

$$\begin{aligned}
a_1 &= \sinh(q)\sinh(2q) & a_2 &= \tfrac{1}{2}(\mathrm{e}^q - \sinh(5q)) \\
a_3 &= \tfrac{1}{2}(\mathrm{e}^{-q} + \sinh(5q)) & a_4 &= \tfrac{1}{4}(\cosh(5q) + 2\cosh(q)) \\
a_5 &= \tfrac{1}{4}(\cosh(5q) - 2\mathrm{e}^{-2q}\cosh(q)) & a_6 &= \tfrac{1}{4}(\cosh(5q) - 2\mathrm{e}^{2q}\cosh(q)) \\
a_7 &= \cosh(q) & a_8 &= \cosh(3q) \\
a_9 &= \mathrm{e}^{2q}\sinh(2q) & a_{10} &= -\mathrm{e}^{-2q}\sinh(2q)
\end{aligned}$$

$$\begin{aligned}
x_1 &= \frac{\mathrm{e}^q}{\cosh(q)} - c\mathrm{e}^{-q} + \tilde{\mu}_-\mu_-\sinh(q)(\mathrm{e}^q - c\cosh(2q)) \\
x_2 &= \frac{c\mathrm{e}^{-q}}{\cosh(q)} + \tilde{\mu}_-\mu_-\cosh(2q)(1 - c\sinh(q))
\end{aligned}$$

$$x_3 = \frac{e^{-3q}}{\cosh(q)} + ce^{-q} - \tilde{\mu}_-\mu_- \sinh(q)(e^{-q} + c\cosh(2q))$$

$$y_1 = c - e^{q} \qquad y_2 = e^{-q}(c + e^{-q}) \qquad z = \cosh(q)$$

$$\rho^k = e^{-q}(c - \sinh(q)) + \tilde{\mu}_-\mu_- \cosh(q)(c\cosh(2q) - \sinh(q))$$

$$\tilde{x}_1 = (a_1 + a_5 + a_6 + 2a_4)x_{11} + (a_1 + a_2 + a_5 + a_6)x_{22} + (a_2 + 2a_5)x_{33}$$

$$\tilde{x}_2 = (a_1 + a_3 + a_5 + a_6)x_{11} + (2a_1 + a_2 + a_3)x_{22} + (a_1 + a_2 + a_5 + a_6)x_{33}$$

$$\tilde{x}_3 = (a_3 + 2a_6)x_{11} + (a_1 + a_3 + a_5 + a_6)x_{22} + (a_1 + 2a_4 + a_5 + a_6)x_{33}$$

$$\tilde{y}_1 = -((c + e^{-5q})a_8 + e^{-3q}(c - e^{5q})a_{10})\sinh(6q)$$

$$\tilde{y}_2 = -((c + e^{-5q})a_9 + e^{-3q}(c - e^{5q})a_8)\sinh(6q)$$

$$\tilde{z} = a_7\cosh(5q)\sinh(6q)$$

$$\mathrm{tr}\,\tilde{k} = x_{11} + x_{22} + x_{33}$$

with

$$x_{11} = \frac{e^{-5q}\sinh(4q)}{\cosh(q)} - ce^{-5q} - \tilde{\mu}_+\mu_+ e^{2q}\cosh(5q)(c\cosh(2q) + e^{-6q}\sinh(q))$$

$$x_{22} = -\frac{e^{-q}}{\cosh(q)}(\sinh(2q) + c\cosh(7q)) - \tilde{\mu}_+\mu_+\cosh(5q)(c\cosh(2q) - \sinh(5q))$$

$$x_{33} = \frac{e^{3q}\sinh(4q)}{\cosh(q)} - ce^{3q} - \tilde{\mu}_+\mu_+ e^{-2q}\cosh(5q)(c\cosh(2q) + e^{6q}\sinh(q)).$$

The spin-1 operator s^3, $s^{\pm}(s^1 \pm is^2)$ is given by

$$s^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad s^+ = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad s^- = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

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Appendix

We now construct the reflecting matrix $K_-(v)$ which can be parametrized as

$$K_-(v) = \rho^K(v)\begin{bmatrix} x_1(v) & y_{11}(v) & z_1(v) \\ y_{21}(v) & x_2(v) & y_{12}(v) \\ z_2(v) & y_{22}(v) & x_3(v) \end{bmatrix} \tag{A1}$$

where $\rho^K(v)$ is an arbitrary function. For simplicity, we denote the (i, j) component of equation (5) as Eq$[i, j]$ at first. One can find that Eq$[j, i]$ can be obtained from Eq$[i, j]$ by interchanging $y_{11} \Leftrightarrow y_{21}$, $y_{12} \Leftrightarrow y_{22}$, $z_1 \Leftrightarrow z_2$ and Eq$[10 - i, 10 - j]$ can be obtained from Eq$[i, j]$ by interchanging $x_1 \Leftrightarrow x_3$, $y_{11} \Leftrightarrow y_{12}$, $y_{21} \Leftrightarrow y_{22}$, $e \Leftrightarrow \bar{e}$, $f \Leftrightarrow \bar{f}$, $g \Leftrightarrow \bar{g}$.

There are 81 function equations and we only pick up some simple-looking ones to obtain the solutions of the matrix $K_-(v)$. Our whole process can be divided into two steps. The first step is to obtain all the non-diagonal elements of $K(v)$. Another step is to achieve the diagonal elements of $K(v)$. We begin with the first step. From Eq[2.8], we have

$$\begin{aligned} &e^{-2q+(u+v)/2}\sinh\big(\tfrac{1}{2}(u-v)\big)y_{11}(u)y_{11}(v)+e^{(v-u)/2}\cosh\big(q-\tfrac{1}{2}(u+v)\big)y_{11}(v)y_{12}(u)\\ &\quad=-e^{2q-(u+v)/2}\sinh\big(\tfrac{1}{2}(u-v)\big)y_{12}(u)y_{12}(v)\\ &\quad\quad+e^{(u-v)/2}\cosh\big(q-\tfrac{1}{2}(u+v)\big)y_{11}(u)y_{12}(v). \end{aligned} \tag{A2}$$

Dividing both side of (A2) by $y_{12}(u)y_{12}(v)$ and differentiating it with respect to u, then letting $u=0$, we have

$$\frac{y_{11}(v)}{y_{12}(v)}=\frac{c-e^{q-v}}{e^{-q}(c+e^{v-q})} \tag{A3}$$

where c is an arbitrary constant. Taking into account that $y_{11}(v)$, $y_{12}(v)$ may be zero, we can obtain

$$y_{11}(v)=\mu(c-e^{q-v})f(v)\qquad y_{12}(v)=\mu e^{-q}(c+e^{v-q})f(v) \tag{A4}$$

where $f(v)\not\equiv 0$ is an arbitrary function. Substituting (A3) into Eq[2.9]

$$\begin{aligned} &\Big(\cosh\big(3q-\tfrac{1}{2}(u+v)\big)\sinh\big(\tfrac{1}{2}(u-v)\big)\\ &\quad-\cosh\big(q-\tfrac{1}{2}(u+v)\big)\sinh\big(-2q+\tfrac{1}{2}(u-v)\big)\Big)y_{12}(v)z_1(u)\\ &\quad=\Big(e^{-2q+(u+v)/2}\sinh(2q)\sinh\big(\tfrac{1}{2}(u-v)\big)y_{11}(u)\\ &\quad\quad+e^{(v-u)/2}\sinh(2q)\cosh\big(q-\tfrac{1}{2}(u+v)\big)y_{12}(u)\Big)z_1(v) \end{aligned} \tag{A5}$$

we obtain

$$\frac{z_1(u)}{z_1(v)}=\frac{\cosh(q-u)f(u)}{\cosh(q-v)f(v)}. \tag{A6}$$

Therefore, we obtain the following result:

$$\begin{aligned} &y_{11}(v)=\mu(c-e^{q-v})f(v)\qquad y_{12}(v)=\mu e^{-q}(c+e^{v-q})f(v)\\ &z_1(v)=\nu\cosh(q-v)f(v) \end{aligned} \tag{A7}$$

where μ, ν are arbitrary constants. Similarly, by Eq[8.2] and Eq[9.2]

$$\begin{aligned} &y_{21}(v)=\tilde{\mu}(\tilde{c}-e^{q-v})g(v)\qquad y_{22}(v)=\tilde{\mu}e^{-q}(\tilde{c}+e^{v-q})g(v)\\ &z_2(v)=\tilde{\nu}\cosh(q-v)g(v) \end{aligned} \tag{A8}$$

where $\tilde{c}$, $\tilde{\mu}$, $\tilde{\nu}$ are arbitrary constants and $g(v)\not\equiv 0$ is an arbitrary function. From Eq[1.1] and Eq[9.9], we have

$$\begin{aligned} &\cosh(3q-(u+v)/2)(y_{11}(u)y_{21}(v)-y_{11}(v)y_{21}(u))\\ &\quad=-(\cosh(3q-(u+v)/2)+e^{q}\sinh((u+v)/2))(z_1(u)z_2(v)-z_1(v)z_2(u)) \end{aligned} \tag{A9}$$

$$\begin{aligned} &\cosh(3q-(u+v)/2)(y_{12}(u)y_{22}(v)-y_{12}(v)y_{22}(u))\\ &\quad=-(\cosh(3q-(u+v)/2)-e^{-q}\sinh((u+v)/2))(z_1(u)z_2(v)-z_1(v)z_2(u)). \end{aligned} \tag{A10}$$

Applying equations (A7) and (A8) to the above two equations, we can obtain the following three results:

$$\begin{aligned} &y_{11}(v)=\mu(c-e^{q-v})g(v)\qquad y_{21}(v)=\tilde{\mu}(c-e^{q-v})g(v)\\ &y_{12}(v)=\mu e^{-q}(c+e^{v-q})g(v)\qquad y_{22}(v)=\tilde{\mu}e^{-q}(c+e^{v-q})g(v)\\ &z_1(v)=\nu\cosh(q-v)g(v)\qquad z_2(v)=\tilde{\nu}\cosh(q-v)g(v) \end{aligned} \tag{A11}$$

$$\begin{aligned} &y_{11}(v) = \mu(c - e^{q-v}) f(v) && y_{21}(v) = 0 \\ &y_{12}(v) = \mu e^{-q}(c + e^{v-q}) f(v) && y_{22}(v) = 0 \\ &z_1(v) = 0 && z_2(v) = \tilde{\nu}\cosh(q - v) g(v) \end{aligned} \tag{A12}$$

$$\begin{aligned} &y_{11}(v) = 0 && y_{21}(v) = \tilde{\mu}(\tilde{c} - e^{q-v}) g(v) \\ &y_{12}(v) = 0 && y_{22}(v) = \tilde{\mu} e^{-q}(\tilde{c} + e^{v-q}) g(v) \\ &z_1(v) = \nu\cosh(q - v) f(v) && z_2(v) = 0 \end{aligned} \tag{A13}$$

where $f(v)/g(v) \not\equiv$ constant. By Eq[1.2], Eq[1.4] and Eq[2.1], Eq[4.1], respectively, we find that both cases (A13) and (A12) do not exist, so there is only one case (A11). Now we deal with the diagonal elements of $K(v)$.

We can see from Eq[1.4] and [4.1] that there are five choices for the parameters $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ as follows:

$$\begin{array}{lllll} \text{(i)} & \mu = 0 & \tilde{\mu} = 0 & & \\ \text{(ii)} & \mu \neq 0 & \tilde{\mu} = 0 & & (\Rightarrow \tilde{\nu} = 0) \\ \text{(iii)} & \mu = 0 & \tilde{\mu} \neq 0 & & (\Rightarrow \nu = 0) \\ \text{(iv.}a) & \mu \neq 0 & \tilde{\mu} \neq 0 & \nu = 0 & (\Rightarrow \tilde{\nu} = 0) \\ \text{(iv.}b) & \mu \neq 0 & \tilde{\mu} \neq 0 & \nu \neq 0 & \left(\Rightarrow \tilde{\nu} = \left(\frac{\tilde{\mu}}{\mu}\right)^2 \nu\right). \end{array} \tag{A14}$$

A.1. Case (i)

For case (i), let $X_i(u) = x_i(u)/[\cosh(q - u)g(u)]$, then from Eq[2,4] and Eq[6,8], we have

$$\frac{e^u X_1(u) e^v X_1(v) - X_2(u) X_2(v) + \nu\tilde{\nu}}{\sinh\left(\frac{1}{2}(u + v)\right)} = \frac{e^u X_1(u) X_2(v) - X_2(u) e^v X_1(v)}{\sinh\left(\frac{1}{2}(u - v)\right)} \tag{A15}$$

$$\frac{e^{-u} X_3(u) e^{-v} X_3(v) - X_2(u) X_2(v) + \nu\tilde{\nu}}{\sinh\left(\frac{1}{2}(u + v)\right)} = \frac{e^{-u} X_3(u) X_2(v) - X_2(u) e^{-v} X_3(v)}{\sinh\left(\frac{1}{2}(u - v)\right)}. \tag{A16}$$

Here we have two choices, (a) $\nu\tilde{\nu} = 0$, (b) $\nu\tilde{\nu} \neq 0$.

A.1.1. Case (a). If $X_2 \equiv 0$, from (A15) and (A16), we can obtain

$$\begin{aligned} &z_1(u) = h(u) && x_1(u) = x_2(u) = x_3(u) = y_{11}(u) = y_{12}(u) = y_{21}(u) = y_{22}(u) \\ & && \quad = z_2(u) = 0 \\ &z_2(u) = h(u) && x_1(u) = x_2(u) = x_3(u) = y_{11}(u) = y_{12}(u) = y_{21}(u) = y_{22}(u) \\ & && \quad = z_1(u) = 0 \end{aligned} \tag{A17}$$

where $h(u) \not\equiv 0$ is an arbitrary function. If $X_2 \not\equiv 0$, following from (A15) and (A16), we have

$$\begin{aligned} x_1(u) &= e^{-u}\left(c_1 e^{u/2} + e^{-u/2}\right)\left(e^{u/2} + c_2 e^{-u/2}\right) f(u) \\ x_2(u) &= \left(e^{u/2} + c_1 e^{-u/2}\right)\left(e^{u/2} + c_2 e^{-u/2}\right) f(u) \\ x_3(u) &= e^u\left(e^{u/2} + c_1 e^{-u/2}\right)\left(c_2 e^{u/2} + e^{-u/2}\right) f(u). \end{aligned} \tag{A18}$$

Here $f(u) \not\equiv 0$ is an arbitrary function, c_1, c_2 are arbitrary parameters. Eq[3.5] implies $c_1 = c_2 = C\mathrm{e}^{-3q}$ with $C^2 = -1$, then we obtain

$$\begin{aligned} x_1(u) &= 2\mathrm{e}^{-3q-u}(\sinh(3q) + C\cosh(u))f(u) \\ x_2(u) &= 2\mathrm{e}^{-3q}(\sinh(3q+u) + C)f(u) \\ x_3(u) &= 2\mathrm{e}^{-3q+u}(\sinh(3q) + C\cosh(u))f(u). \end{aligned} \tag{A19}$$

The $\nu\tilde{\nu} = 0$ contain the following three cases:

$$\begin{array}{lll} \text{(a1)} & \nu = 0 & \tilde{\nu} = 0 \\ \text{(a2)} & \nu \neq 0 & \tilde{\nu} = 0 \\ \text{(a3)} & \nu = 0 & \tilde{\nu} \neq 0. \end{array} \tag{A20}$$

For the case (a1), this is a diagonal solution

$$\begin{aligned} x_1(u) &= \mathrm{e}^{-u}\left(\cosh\left(3q - \tfrac{1}{2}u\right) + C\sinh\left(\tfrac{1}{2}u\right)\right)\tilde{\rho}^k(u) \\ x_2(u) &= \left(\cosh\left(3q + \tfrac{1}{2}u\right) - C\sinh\left(\tfrac{1}{2}u\right)\right)\tilde{\rho}^k(u) \\ x_3(u) &= \mathrm{e}^{u}\left(\cosh\left(3q - \tfrac{1}{2}u\right) + C\sinh\left(\tfrac{1}{2}u\right)\right)\tilde{\rho}^k(u) \end{aligned} \tag{A21}$$

with $\tilde{\rho}^k(u) = 2\mathrm{e}^{-3q}(\sinh(3q+u/2)+C\cosh(u/2))(\cosh(3q))^{-1}f(u)$. For case (a2), by Eq[2,6] and Eq[1,3] we find that this case does not exist. For case (a3), by Eq[6,2] and Eq[3,1] we find that this case also does not exist.

A.1.2. Case (b). When $v = 0$ in (A15), we can find that $\mathrm{e}^u X_1(u) + X_2(u)$ is a constant and $\mathrm{e}^u X_1(u) - X_2(u)$ is also a constant while $v = \sqrt{-1}\pi$ in (A15). Therefore, $\mathrm{e}^u X_1(u)$ and $X_2(u)$ are both constants, $\mathrm{e}^u X_1(u) = c_1$, $X_2(u) = c_2$ with $c_1^2 - c_2^2 + \nu\tilde{\nu} = 0$. Similarly, from (A16) we have $\mathrm{e}^{-u}X_3(u) = c_3$ with $c_3^2 - c_2^2 + \nu\tilde{\nu} = 0$. We now have two possibilities, $c_1 = c_3$ or $c_1 = -c_3$. When $c_1 = c_3$, Eq[2,6] implies $c_1 = 0$ and Eq[1,3] indicates $c_2 = 0$, which contradicts $\nu\tilde{\nu} \neq 0$. When $c_1 = -c_3$, Eq[3,7] implies $c_1 = 0$ and Eq[1,3] indicates $c_2 = 0$, which also contradicts $\nu\tilde{\nu} \neq 0$. Therefore, there is only diagonal solution for case (i).

When $g(0) = 0$, the above discussion does not work. The solutions for this case are obtained in [10], which have two free parameters and will reduce to the trivial diagonal solutions when one of the parameters vanishes.

A.2. Case (ii)

By Eq[1,4] we obtain

$$x_1(u) = \frac{\mathrm{e}^{-q}}{\sinh(u)}(c_1 + c_2\mathrm{e}^{-u})(c - \mathrm{e}^{q-u})g(u) \tag{A22}$$

$$x_2(u) = \frac{\mathrm{e}^{-q}}{\sinh(u)}(c_1 + c_2\mathrm{e}^{u})(c - \mathrm{e}^{q-u})g(u) \tag{A23}$$

where c_1 and c_2 are arbitrary constants. By Eq[6,9] we obtain

$$x_3(u) = \frac{\mathrm{e}^{2q}}{\sinh(u)}(c_3 + c_4\mathrm{e}^{u})\mathrm{e}^{-q}(c + \mathrm{e}^{u-q})g(u) \tag{A24}$$

$$x_2(u) = \frac{\mathrm{e}^{2q}}{\sinh(u)}(c_3 + c_4\mathrm{e}^{-u})\mathrm{e}^{-q}(c - \mathrm{e}^{u-q})g(u) \tag{A25}$$

where c_3 and c_4 are arbitrary constants. From (A23) and (A25), we have

$$c_1 = -cc_4 \mathrm{e}^{q} \tag{A26}$$

$$c_3 = cc_2 \mathrm{e}^{-q} \tag{A27}$$

$$(1 + c^2)(c_2 + c_4) = 0. \tag{A28}$$

From (A28) we have two choices $c_2 = -c_4$ or $c^2 = -1$. When $c_2 = -c_4$, Eq[2,6] implies that c_2 and c_4 are not constants which contradicts (A23) and (A25). When $c^2 = -1$, substituting (A22), (A23) and (A27) into Eq[2,6], we obtain

$$c_1 = -\frac{\mu^2}{\nu} \frac{c\mathrm{e}^{-2q} \sinh(q)}{\sinh(2q)} \tag{A29}$$

$$c_2 = \frac{\mu^2}{\nu} \frac{\mathrm{e}^{q} \sinh(q)}{\sinh(2q)}. \tag{A30}$$

Then we obtain

$$\begin{aligned}
x_1(u) &= \frac{1}{\sinh(u)} \frac{\mu^2}{\nu} \frac{\mathrm{e}^{-q-u}}{\cosh(q)} (c \cosh(q) + \sinh(u - 2q))g(u) \\
x_2(u) &= \frac{1}{\sinh(u)} \frac{\mu^2}{\nu} \frac{\mathrm{e}^{-q}}{\cosh(q)} (c \cosh(q + u) - \sinh(2q))g(u) \\
x_3(u) &= \frac{1}{\sinh(u)} \frac{\mu^2}{\nu} \frac{\mathrm{e}^{-q+u}}{\cosh(q)} (c \cosh(q) + \sinh(u - 2q))g(u)
\end{aligned} \tag{A31}$$

with

$$\begin{aligned}
&y_{11}(u) = \mu(c - \mathrm{e}^{q-u})g(u) \qquad y_{12}(u) = \mu \mathrm{e}^{-q}(c + \mathrm{e}^{u-q})g(u) \\
&z_1(u) = \nu \cosh(q - u)g(u) \qquad y_{21}(u) = y_{22}(u) = z_2(u) = 0.
\end{aligned} \tag{A32}$$

A.3. Case (iii)

Like case (ii), by Eq[4,1], Eq[9,6] and Eq[6,2] we can obtain

$$\begin{aligned}
&y_{21}(u) = \tilde{\mu}(c - \mathrm{e}^{q-u})g(u) \qquad y_{22}(u) = \tilde{\mu} \mathrm{e}^{-q}(c + \mathrm{e}^{u-q})g(u) \\
&z_2(u) = \tilde{\nu} \cosh(q - u)g(u) \qquad y_{11}(u) = y_{12}(u) = z_1(u) = 0 \\
&x_1(u) = \frac{1}{\sinh(u)} \frac{\tilde{\mu}^2}{\tilde{\nu}} \frac{\mathrm{e}^{-q-u}}{\cosh(q)} (c \cosh(q) + \sinh(u - 2q))g(u) \\
&x_2(u) = \frac{1}{\sinh(u)} \frac{\tilde{\mu}^2}{\tilde{\nu}} \frac{\mathrm{e}^{-q}}{\cosh(q)} (c \cosh(q + u) - \sinh(2q))g(u) \\
&x_3(u) = \frac{1}{\sinh(u)} \frac{\tilde{\mu}^2}{\tilde{\nu}} \frac{\mathrm{e}^{-q+u}}{\cosh(q)} (c \cosh(q) + \sinh(u - 2q))g(u)
\end{aligned} \tag{A33}$$

with $c^2 = -1$.

A.4. Case (iv.a)

Eq[1,7] implies $\mu = 0$ and Eq[7,1] implies $\tilde{\mu} = 0$ which contradicts our assumption $\mu \neq 0$ and $\tilde{\mu} \neq 0$. Therefore, this case does not exist.

A.5. Case (iv.b)

By Eq[1,4] we obtain

$$x_1(v) = \frac{e^{-q}}{\sinh(v)}\left[\left(c_1 + c_2e^{-v}\right)\left(c - e^{q-v}\right) + \frac{\tilde{\mu}\nu}{2\mu}\left(ce^{q-v}\cosh(q) + ce^{v}\cosh(2q)\right.\right.$$
$$\left.\left. - e^{q-2v}\sinh(2q) - \sinh(3q)\right)\right]g(v) \tag{A34}$$

where c_1 and c_2 are arbitrary constants. Substituting $x_1(v)$ into Eq[1.4], we obtain

$$\begin{aligned} x_2(u) = \frac{e^{-q}}{\sinh(u)}\Big[&(c_1 + c_2e^{u})(c - e^{q-u}) \\ &+ \frac{\tilde{\mu}\nu}{2\mu}(ce^{q-u}\cosh(q) + ce^{u}\cosh(2q) - e^{q}\sinh(2q) - \sinh(3q))\Big]g(u) \\ &+ \frac{\tilde{\mu}\nu}{\mu}(c\cosh(q) + \cosh(u-2q) \\ &+ \frac{e^{-q} + ce^{-v} - (1+c^2)e^{u+v-q}}{c - e^{q-v}}\cosh(2q))g(u). \end{aligned} \tag{A35}$$

We know that $x_2(u)$ should not depend on v. It is easy to find that only when $c^2 = -1$ can v disappear from $x_2(u)$, therefore we obtain

$$\begin{aligned} x_2(u) = \frac{e^{-q}}{\sinh(u)}\Big[&(c_1 + c_2e^{u})(c - e^{q-u}) \\ &+ \frac{\tilde{\mu}\nu}{2\mu}(ce^{q-u}\cosh(q) + ce^{u}\cosh(2q) - e^{q}\sinh(2q) - \sinh(3q))\Big]g(u) \\ &+ \frac{\tilde{\mu}\nu}{\mu}(ce^{-2q}\sinh(q) + \cosh(u-2q))g(u). \end{aligned} \tag{A36}$$

Similarly, by Eq[6.9]

$$x_3(u) = \frac{e^{2q}}{\sinh(u)}\left[(c_3 + c_4e^{u})e^{-q}(c + e^{u-q}) + \frac{\tilde{\mu}\nu}{2\mu}e^{-2q}(ce^{q-u}\cosh(2q) + ce^{u}\cosh(q)\right.$$
$$\left. - e^{2u}\sinh(2q) - e^{q}\sinh(3q))\right]g(u) \tag{A37}$$

$$\begin{aligned} x_2(u) = \frac{e^{2q}}{\sinh(u)}\Big[&(c_3 + c_4e^{-u})e^{-q}(c + e^{u-q}) \\ &+ \frac{\tilde{\mu}\nu}{2\mu}e^{-2q}(ce^{q-u}\cosh(2q) + ce^{u}\cosh(q) - \sinh(2q) - e^{q}\sinh(3q))\Big]g(u) \\ &+ \frac{\tilde{\mu}\nu}{\mu}(ce^{2q}\sinh(q) + \cosh(u-2q))g(u) \end{aligned} \tag{A38}$$

with $c^2 = -1$, where c_3 and c_4 are arbitrary constants. Combining equations (A36) and (A38), there are

$$\begin{aligned} c_3 &= cc_2e^{-q} - \frac{c}{2}\frac{\tilde{\mu}\nu}{\mu}e^{2q}\sinh(q) \\ c_4 &= cc_1e^{-q} - \frac{1}{2}\frac{\tilde{\mu}\nu}{\mu}e^{-3q}\sinh(q). \end{aligned} \tag{A39}$$

So $x_3(u)$ can be rewritten as

$$x_3(u) = \frac{e^{2q}}{\sinh(u)}\Bigg[(cc_2 + cc_1 e^{u})e^{-2q}(c + e^{u-q}) - \frac{\tilde{\mu}\nu}{2\mu}\sinh(q)(ce^{2q} + e^{u-3q})e^{-q}(c + e^{u-q}) + \frac{\tilde{\mu}\nu}{2\mu}e^{-2q}(ce^{q-u}\cosh(2q) + ce^{u}\cosh(q) - e^{2u}\sinh(2q) - e^{q}\sinh(3q))\Bigg]g(u). \tag{A40}$$

The two constants c_1 and c_2 remain unknown. Substituting equations (A34) and (A36) into Eq[2.4], we obtain

$$\begin{aligned} c_1 &= -c\frac{\mu^2}{\nu}\frac{e^{-2q}\sinh(q)}{\sinh(2q)} - c\frac{\tilde{\mu}\nu}{\mu}\sinh(q)\cosh(2q) \\ c_2 &= \frac{\mu^2}{\nu}\frac{e^{q}\sinh(q)}{\sinh(2q)} - \frac{\tilde{\mu}\nu}{\mu}e^{-q}\sinh(q). \end{aligned} \tag{A41}$$

Substituting equation (A41) into equations (A34), (A36) and (A40), we achieve the final results,

$$\begin{aligned} x_1(u) &= \frac{1}{\sinh(u)}\Bigg[\frac{\mu^2}{\nu}\frac{e^{-q-u}}{\cosh(q)}(c\cosh(q) + \sinh(u-2q)) + \frac{\tilde{\mu}\nu}{\mu}\cosh(q-u)(c\cosh(2q) - e^{-u}\sinh(q))\Bigg]g(u) \\ x_2(u) &= \frac{1}{\sinh(u)}\Bigg[\frac{\mu^2}{\nu}\frac{e^{-q}}{\cosh(q)}(c\cosh(q+u) - \sinh(2q)) + \frac{\tilde{\mu}\nu}{\mu}\cosh(q-u)(c\cosh(2q) + \sinh(u-q))\Bigg]g(u) \\ x_3(u) &= \frac{1}{\sinh(u)}\Bigg[\frac{\mu^2}{\nu}\frac{e^{-q+u}}{\cosh(q)}(c\cosh(q) + \sinh(u-2q)) + \frac{\tilde{\mu}\nu}{\mu}\cosh(q-u)(c\cosh(2q) - e^{u}\sinh(q))\Bigg]g(u). \end{aligned} \tag{A42}$$

$$\begin{aligned} y_{11}(u) &= \mu(c - e^{q-u})g(u) & y_{21}(u) &= \tilde{\mu}(c - e^{q-u})g(u) \\ y_{12}(u) &= \mu e^{-q}(c + e^{u-q})g(u) & y_{22}(u) &= \tilde{\mu}e^{-q}(c + e^{u-q})g(u) \\ z_1(u) &= \nu\cosh(q-u)g(u) & z_2(u) &= \tilde{\nu}\cosh(q-u)g(u) \end{aligned} \tag{A43}$$

with $c^2 = -1$.

A.6. Summary

Letting $\nu = \mu^2$, $\tilde{\nu} = \tilde{\mu}^2$ and

$$g(u) = \frac{e^{q}\sinh(u)\cosh(3q)}{\sinh(u/2 - 2q) + c\cosh(q + u/2)}\tilde{\rho}^{k}(u)$$

in (A42) and (A43), then taking $\mu = 0$, $\tilde{\mu} = 0$, we can obtain (A21). Taking $\tilde{\mu} = \tilde{\nu} = 0$ in (A42) and (A43), we can obtain (A31) and (A32). The (A33) can be obtained by replacing the ν in (A42) and (A43) with $(\mu/\tilde{\mu})^2\tilde{\nu}$ and taking $\mu = 0$.

We have checked Eq$[i, j](i, j \in [1, 9])$ by substituting (A42) and (A43) into (5).

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